NUMERICAL ANALYSIS TOPIC IV
POLYNOMIAL INTERPOLATION

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1. Introduction

1.1. **Tables.** Let $T = \{(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)\}$ be a collection of ordered pairs of real numbers, with the property that $x_i = x_j \Rightarrow i = j$. We think of T as a table of data. A *polynomial interpolation* of T is a polynomial $p(x)$ such that $p(x_i) = y_i$ for $i = 0, 1, \dots, n$.

If $f : \mathbb{R} \to \mathbb{R}$ is a function, we may select points $x_0, \dots, x_n$ in the domain of f and set $T = \{(x_i, f(x_i)\}$; the polynomial that interpolates T is called an interpolation of f, and may approximate f in some manner, just as the secant line through to nearby points on the graph of a differentiable function is an approximation for the function near those points. The values x_i are called the *nodes* of the interpolation.

Our goal is two find a polynomial of minimal degree which interpolates a given table. There are two approaches.

1.2. **Lagrange Approach.** The Lagrange approach considers the entire table simultaneously. The idea is to construct polynomial functions which are basis elements for the vector space of degree n polynomials which are nonzero at some $x \in \{x_0, \dots, x_n\}$. These functions are equal to 1 at one of the x_i's, and are equal to 0 at every x_j with $j \neq i$, and are called *cardinal functions*. We can take a linear combination of these cardinal functions to find the interpolation polynomial.

1.3. **Newton Approach.** The Newton approach inductively builds the polynomial one point at a time. Here the idea is to start with a polynomial p_0 of degree zero and add to it a polynomial of degree one which equals zero at x_0 and equals y_1 at x_1. Call this polynomial p_1. Now add to it a degree two polynomial which equals zero at x_0 and x_1, but equals y_2 at x_2; call it p_2. Now p_2 interpolates the first three points. Continue in this fashion.

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2. Lagrange Form

Proposition 1. *Let $T = \{(x_0, y_0), \dots, (x_n, y_n)\} \subset \mathbb{R}^2$ be a table. Then there exists a unique polynomial $p(x)$ of degree n such that $p(x_i) = y_i$ for $i = 0, 1, \dots, n$.*

Proof. The i^{th} *cardinal function* of T is defined to be

$$\ell_i(x) = \prod_{j=0 (j \neq i)}^{n} \left(\frac{x - x_j}{x_i - x_j} \right).$$

In expanded form, one sees that this function is a product of n linear factors:

$$\ell_i(x) = \left(\frac{x - x_0}{x_i - x_0} \right) \left(\frac{x - x_1}{x_i - x_1} \right) \cdots \left(\frac{x - x_{i-1}}{x_i - x_{i-1}} \right) \left(\frac{x - x_{i+1}}{x_i - x_{i+1}} \right) \cdots \left(\frac{x - x_n}{x_i - x_n} \right);$$

the denominators are constant. Thus the degree of $\ell_i(x)$ is n. Note that $\ell_i(x)$ has the property that

$$\ell_i(x) = \begin{cases} 1 & \text{if } x = x_i; \\ 0 & \text{if } x = x_j \text{ for some } j \neq i. \end{cases}$$

Now set

$$p(x) = \sum_{i=0}^{n} y_i \ell_i(x).$$

Then $p(x_i) = y_i$ for $i = 0, 1, \dots, n$. Moreover, $\deg(p) \leq n$, since $p(x)$ is the sum of polynomials of degree no more than n.

As for uniqueness, suppose that $q(x)$ is another polynomial satisfying $q(x_i) = y_i$ for $i = 0, 1, \dots, n$. Then $p(x) - q(x)$ is a polynomial of degree at most n with roots at $x_0, \dots, x_n$; that is, it has $n + 1$ roots. Thus it must be the zero polynomial. □

3. Newton Form

Proposition 2. *Let $T = \{(x_0, y_0), \dots, (x_n, y_n)\} \subset \mathbb{R}^2$ be a table. If $p_{n-1}(x)$ interpolates $\{(x_0, y_0), \dots, (x_{n-1}, y_{n-1})\}$, then $p_n(x)$ interpolates $\{(x_0, y_0), \dots, (x_n, y_n)\}$, where*

$$p_n(x) = p_{n-1}(x) + c_i \prod_{i=0}^{n-1} (x - x_i),$$

and c_i is determined by $p_n(x_n) = y_n$.

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